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A THEOREM ON BINARY DIGITS

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Let $B(n)$ denote the number of digits 1 in the representation of a natural number n in the binary scale. It is well known that for most n , the number $B(n)$ is about half the total number of digits, so that $B(n)$ is roughly equal to $\frac{1}{2}v$, where $v = v(n) = \log_2 n$, with \log_2 denoting the logarithm to the base 2. In fact it follows from the Central Limit Theorem of probability theory that the numbers n with

$$\frac{B(n) - \frac{1}{2}v}{\sqrt{v}} \leq \xi$$

have density

$$\varphi(\xi) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\xi} e^{-2t^2} dt.$$

Here we say that a set S of natural numbers has density φ if the number $S(x)$ of elements $n \in S$, $n \leq x$ satisfies the asymptotic relation $S(x) \sim \varphi x$ as $x \rightarrow \infty$.

Stolursky was the first to compare $B(n)$ and $B(kn)$ where $k > 1$ is a fixed odd integer. He called a number k -sturdy if $B(n) \leq B(kn)$,

and simply sturdy if it is k -sturdy for every k . Stolarsky proved that the 3-sturdy numbers have density $\frac{1}{2}$. Here we are going to sketch a proof that for any odd $k > 1$, the k -sturdy numbers have density $\frac{1}{2}$. The interest in the proof lies in the fact that it uses Markov chains. The main result is as follows.

THEOREM. Let k_1, \dots, k_s be distinct odd integers. The matrix $M = (m_{ij})$ with entries $m_{ij} = k_i^{-1} k_j^{-1} (\gcd(k_i, k_j))^2$ ($1 \leq i, j \leq s$) has an inverse $Q = (q_{ij})$, and the quadratic form $Q(\underline{t}) = Q(t_1, \dots, t_s) = \sum_{i,j=1}^s q_{ij} t_i t_j$ is positive definite. Hence

$$\varphi(\xi_1, \dots, \xi_s) = (2/\pi)^{s/2} (\det M)^{-1/2} \int_{-\infty}^{\xi_1} \dots \int_{-\infty}^{\xi_s} dt_1 \dots dt_s$$

is well defined for $(\xi_1, \dots, \xi_s) \in \mathbb{R}^s$. The main assertion now is that the natural numbers n having simultaneously

$$\frac{B(k_i n) - \frac{1}{2} \nu}{\sqrt{\nu}} \leq \xi_i \quad (i=1, \dots, s)$$

have density $\varphi(\xi_1, \dots, \xi_s)$.

A corollary is that for distinct odd integers k_1, k_2 , the numbers n with $B(k_1 n) - B(k_2 n) \leq \eta \sqrt{\nu}$ have density

$$\varphi(\eta) = (\pi(1-\mu))^{-1/2} \int_{-\infty}^{\eta} e^{-t^2/(1-\mu)} dt$$

with $\mu = k_1^{-1} k_2^{-1} (\gcd(k_1, k_2))^2$. In particular, numbers n with

$B(k_1 n) \leq B(k_2 n)$ have density $\varphi(0) = \frac{1}{2}$. Another corollary is that

the sturdy numbers have density 0.

Let \mathbb{Z}_2 be the ring of 2-adic integers

$$N = a_1 + 2a_2 + 2^2a_3 + \dots,$$

with each "digit" a_t either 0 or 1. The triple $(\Omega, \mathcal{F}, \underline{P})$, where \underline{P} is the Haar measure on Ω , and \mathcal{F} consists of \underline{P} -measurable subsets of Ω , is a probability triple. Write $B_h(N)$ for the number of digits 1 among a_1, \dots, a_h . Given distinct odd k_1, \dots, k_s , put

$$S_h^{(i)} = B_h(k_i N) - \frac{1}{2}h \quad (i=1, \dots, s), \text{ and write}$$

$$\underline{R}_h(\underline{f}_1, \dots, \underline{f}_s) = \underline{P} \left\{ h^{-1/2} S_h^{(i)} \leq \underline{f}_i \quad (i=1, \dots, s) \right\}.$$

The theorem can be shown to be a consequence of the

PROPOSITION. $\lim_{h \rightarrow \infty} \underline{R}_h(\underline{f}_1, \dots, \underline{f}_s) = \mathcal{G}(\underline{f}_1, \dots, \underline{f}_s).$

Write $k_i N = b_1^{(i)} + 2b_2^{(i)} + \dots$, and put $\gamma_t^{(i)} = \begin{cases} \frac{1}{2} & \text{if } b_t^{(i)} = 1 \\ -\frac{1}{2} & \text{if } b_t^{(i)} = 0. \end{cases}$

Then $S_h^{(i)} = \gamma_1^{(i)} + \dots + \gamma_h^{(i)}$. For given i , the random variables $\gamma_1^{(i)}, \gamma_2^{(i)}, \dots$ are independent. But the random vectors $\underline{\gamma}_1, \underline{\gamma}_2, \dots$, where $\underline{\gamma}_t = (\gamma_t^{(1)}, \dots, \gamma_t^{(s)})$, are not independent. Hence we cannot use the Central Limit Theorem for sums of independent random variables.

Call a vector $\underline{x} = (x_1, \dots, x_s)$ admissible, if there is a real number α in $0 \leq \alpha < 2$ with $x_i = [k_i \alpha]$ ($i=1, \dots, s$), where $[]$ denotes the integer part. There are finitely many admissible vectors. The vector $\underline{0} = (0, \dots, 0)$ is admissible, and if \underline{x} is admissible, then both $\underline{x}^0 = ([x_1/2], \dots, [x_s/2])$ and $\underline{x}^1 = ([x_1/2] + k_1, \dots, [x_s/2] + k_s)$ are admissible.

Given N as above, put $N_t = a_1 + 2a_2 + \dots + 2^{t-1}a_t$. Put $\underline{x}_0 = \underline{0}$

and $\underline{x}_t = \underline{x}_t(\mathcal{N}) = ([2^{-t+1} k_1, \eta_t], \dots, [2^{-t+1} k_s, \eta_t]) \quad (t=1, 2, \dots)$.

Then $\underline{x}_0, \underline{x}_1, \dots$ are (vector valued) random variables. Clearly $\underline{x}_t(\mathcal{N})$ is always admissible, and conversely if \underline{x} is admissible, then there is an \mathcal{N} and a t with $\underline{x}_t(\mathcal{N}) = \underline{x}$. It is now easy to prove the

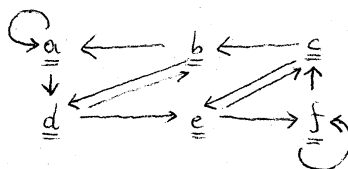
LEMMA. The random variables $\underline{x}_0, \underline{x}_1, \dots$ form a Markov chain.

The transition probabilities are given by the rule that for given \underline{x}_t , we have \underline{x}_{t+1} either equal to \underline{x}_t^0 , or to \underline{x}_t^1 , each with probability $\frac{1}{2}$.

We now observe that the random variable \underline{y}_t is a "functional" of \underline{x}_t : We have $\underline{y}_t^{(i)} = f^{(i)}(\underline{x}_t)$, where $f^{(i)}(\underline{x}) = \frac{1}{2}$ if x_i is odd, $f^{(i)}(\underline{x}) = -\frac{1}{2}$ if x_i is even. Hence the machinery of Markov Chains can be used to complete the proof of our theorem.

In the special case when $s=2$, $k_1=1$, $k_2=3$, there are 6 admissible vectors: $\underline{a}=(0,0)$, $\underline{b}=(0,1)$, $\underline{c}=(0,2)$, $\underline{d}=(1,3)$, $\underline{e}=(1,4)$, $\underline{f}=(1,5)$.

The transition probabilities are given by the following diagram, where each arrow represents a probability of $\frac{1}{2}$:



References

[Hung. (To appear)]

W.M. Schmidt. The Joint Distribution of the digits of certain integer s -tuples. Acta Math.

K.B. Stolarsky. Integers whose multiples have anomalous digital frequencies. Acta Arith. (To appear).